Topological Hochschild homology

Usual definition. Topological Hochschild homology is defined as the realization of the cyclic bar construction:

$$THH(A) = |\mathrm{Bar}_{\mathrm{cyc}}(A)| = \left| \mathrm{Bar}(A) \underset{A \otimes A^{\mathrm{op}}}{\otimes} A \right| \tag{1}$$

for an associative (cofibrant) monoid A.

A category of configurations in S^1. The usual pictorial interpretation of the cyclic bar construction using points on a circle can be made precise. For that purpose we define the Top-enriched category $\underline{\mathsf{M}}(S^1)$:

- The objects of $\underline{\mathsf{M}}(S^1)$ are the finite subsets of S^1.
- Given $S, T \subset S^1$ two finite subsets of S^1, the topological space of morphisms $\underline{\mathsf{M}}(S^1)(S,T)$ is the space of triples (f, t, α) in which $f : S \to T$ is a function, $t \in [0, +\infty[$ and $\alpha : S \times [0,t] \to S^1$ is a homotopy from the inclusion $S \hookrightarrow S^1$ to the composition $S \underset{f}{\to} T \hookrightarrow S^1$ verifying

$$\alpha(x,u) = \alpha(y,u) \Rightarrow \underset{v \in [u,t]}{\forall} \alpha(x,v) = \alpha(y,v)$$

  for all $x, y \in S$, $u \in [0,t]$

Let $\mathrm{Ord}\mathbb{Z}$ be the category obtained by taking connected components of the morphism spaces in $\underline{\mathsf{M}}(S^1)$. Then the functor $\underline{\mathsf{M}}(S^1) \to \mathrm{Ord}\mathbb{Z}$ is a bijection on objects and a weak equivalence on morphism spaces. An elegant alternative description of $\mathrm{Ord}\mathbb{Z}$ is given by Elmendorf [2].

THH from $\mathrm{Ord}\mathbb{Z}$. In order to connect back to Hochschild homology, observe that there exists a functor $\Delta^{\mathrm{op}} \to \mathrm{Ord}\mathbb{Z}$ and $\mathrm{Bar}_{\mathrm{cyc}}(A)$ extends to

$$hh(A) : \mathrm{Ord}\mathbb{Z} \to C$$

The methods of [1] prove that $\Delta^{\mathrm{op}} \to \mathrm{Ord}\mathbb{Z}$ is homotopy cofinal and so topological Hochschild homology can be recovered as a homotopy colimit over $\mathrm{Ord}\mathbb{Z}$:

$$THH(A) \simeq \underset{\mathrm{Ord}\mathbb{Z}}{\mathrm{hocolim}}\ hh(A)$$

To be more precise on the proof of the mentioned homotopy cofinality, let Z be the left Kan extension of $\Delta^\bullet : \Delta \to s\mathrm{Set}$ along $\Delta^{\mathrm{op}} \to \mathrm{Ord}\mathbb{Z}$. Then using the duality $\mathrm{Ord}\mathbb{Z} \simeq \mathrm{Ord}\mathbb{Z}^{\mathrm{op}}$ (given in [2]), Z can be seen to be:

$$\begin{aligned} Z(k)[l] &= \mathrm{Ord}\mathbb{Z}(l+1, k) \\ &= \{((t, i_{j+1}), \ldots, (t, i_l), (t+1, i_0), \ldots, (t+1, i_j)) : t \in \mathbb{Z}, 0 \le j \le l, \\ &\qquad\qquad\qquad\qquad 0 \le i_0 \le \ldots \le i_l \le k\} \end{aligned}$$

with the obvious faces and degeneracies. This gives a triangulation of $\mathbb{R} \times \Delta^k$ and so $|Z(k)| = \mathbb{R} \times \Delta^k \simeq *$.

Associative monoids

Classifying associative monoids. $\mathrm{Ord}\Sigma$ is the category of finite non-commutative sets. Its objects are finite sets and an element of $\mathrm{Ord}\Sigma(S,T)$ is a map of finite sets $f : S \to T$ and a total order in $f^{-1}(x)$ for each $x \in T$. There is a symmetric monoidal structure on $\mathrm{Ord}\Sigma$ that on objects is given by the disjoint union of finite sets. The category $\mathrm{Ord}\Sigma$ classifies associative monoids in a symmetric monoidal category. More precisely, the category of associative monoids in a symmetric monoidal category C is equivalent to the category of symmetric monoidal functors $\mathrm{Ord}\Sigma \to C$.

More generally, given on operad P, there exists a symmetric monoidal category $\underline{P}$ such that the category of P-algebras in a symmetric monoidal category C is equivalent to the category of symmetric monoidal functors $\underline{P} \to C$. The category $\underline{P}$ has $\mathbb{N}$ as its set of objects, where the symmetric monoidal structure is given by addition in $\mathbb{N}$. Furthermore, there is a symmetric monoidal functor $\underline{P} \to (\mathrm{FinSet}, \amalg)$ and $P(n) = \underline{P}(n, 1)$ for $n \in \mathbb{N}$. The conclusion of the previous paragraph can then be restated as $\underline{\mathsf{Ass}} \simeq \mathrm{Ord}\Sigma$.

THH from OrdΣ. There is a functor

$$i : \Delta^{\mathrm{op}} \to \mathrm{Ord}\Sigma$$

$$i[n] = n + 1$$

We can compute the left Kan extension of $\Delta^\bullet : \Delta \to s\mathrm{Set}$ along i^{op}, which we call X:

$$X(k) = (\mathsf{LKE}_{i^{\mathrm{op}}} \Delta^\bullet)(k) = \mathrm{Ord}\Sigma(k, i-)$$

Then based on the argument of [1] we can show that

$$X(k) \simeq (S^1)^{\amalg (k-1)!} \simeq S^1 \times (\Sigma_k / C_k)$$

which was proved in [6]. The relevance of these spaces is that

$$THH(A) = X \underset{\mathrm{Ord}\Sigma}{\otimes} \underline{A} \tag{2}$$

for any associative monoid A. $\underline{A}$ denotes the symmetric monoidal functor $\mathrm{Ord}\Sigma \to C$ that classifies A. We can also relate X to $\mathrm{Ord}\mathbb{Z}$: there is a natural functor $\mathrm{Ord}\mathbb{Z} \to \mathrm{Ord}\Sigma$ which is (essentially) a category fibred in groupoids. The corresponding fibre functor $Y : \mathrm{Ord}\Sigma^{\mathrm{op}} \to \mathrm{Grpd}$ has a nerve equivalent to X.

Configuration spaces of S^1. The first observation we can make is that

$$|X(k)| \simeq S^1 \times (\Sigma_k / C_k) \simeq \mathsf{Conf}(S^1, k)$$

Can we formulate the functor X in a natural way? Well, for that purpose, let us look at an equivalent replacement of the category $\underline{\mathsf{Ass}}$, namely $\underline{\mathsf{D}_1}$ where D_1 is the little intervals operad. The equivalence $\mathsf{D}_1 \to \mathsf{Ass}$ gives us an equivalence $\underline{D_1} \to \underline{\mathsf{Ass}}$. A description of $\underline{\mathsf{D}_1}$ is:

$$\begin{aligned} \underline{\mathsf{D}_1}(k, l) = \{ f \in \mathrm{Emb}([0,1]^{\amalg k}, [0,1]^{\amalg l}) : & f \text{ preserves orientation,} \\ & f \text{ has locally constant speed} \} \end{aligned}$$

and the functor $\underline{\mathsf{D}_1}^{\mathrm{op}} \to \underline{\mathsf{Ass}}^{\mathrm{op}} \xrightarrow{|X|} \mathrm{Top}$ is equivalent to $\mathsf{D}[S^1]$ given by:

$$\begin{aligned} \mathsf{D}[S^1](k) = \{ f \in \mathrm{Emb}([0,1]^{\amalg k}, S^1) : & f \text{ preserves orientation,} \\ & f \text{ has locally constant speed} \} \end{aligned}$$

Note that there is a canonical equivalence $D[S^1](k) \xrightarrow{\sim} \mathsf{Conf}(S^1, k)$ for $k \in \mathbb{N}$. Note also that the conditions on locally constant speed don't actually change the homotopy type and therefore are not really necessary. In any event:

$$\mathsf{D}[S^1] \underset{\underline{\mathsf{D}_1}}{\otimes} \underline{A} \simeq X \underset{\mathrm{Ord}\Sigma}{\otimes} \underline{A} = THH(A)$$

Commutative monoids

Classifying commutative monoids. The commutative operad Comm is such that $\underline{\mathsf{Comm}} \simeq (\mathrm{FinSet}, \amalg)$. Therefore a commutative monoid in a symmetric monoidal category is the same as a symmetric monoidal functor $\mathrm{FinSet} \to C$.

THH **from FinSet.** Let $j : \Delta^{\mathrm{op}} \to \mathrm{FinSet}$ be the composite

$$\Delta^{\mathrm{op}} \xrightarrow{i} \mathrm{Ord}\Sigma \to \mathrm{FinSet}$$

Then the left Kan extension of $\Delta^\bullet$ along j^{op} is:

$$\left(\mathsf{LKE}_{j^{\mathrm{op}}}\Delta^\bullet\right)(k) = \mathrm{FinSet}(k, j-)$$

Noting that $S^1 = \Delta^1/\partial\Delta^1$ has $l+1$ l-simplices and that $\mathrm{FinSet}(k, j([l]))$ has $(l+1)^k$ elements it becomes straightforward to prove

$$\mathrm{FinSet}(k, j-) = (S^1)^{\times k}$$

and thus

$$\mathsf{LKE}_{j^{\mathrm{op}}}\Delta^\bullet = (S^1)^{\times -} = s\mathrm{Set}(-, S^1)$$

In particular

$$THH(A) = (S^1)^{\times -} \underset{\mathrm{FinSet}}{\otimes} \underline{A} \tag{3}$$

for a commutative monoid A. This expression can be seen to coincide with $S^1 \otimes A$ (see [3]), the tensor of S^1 with the commutative monoid A coming from the enrichment of commutative monoids over sSet.

Spectral sequences

Bökstedt spectral sequence. The definition (1) gives us a spectral sequence for an associative ring spectrum A:

$$HH_*(H_*(A)) \Longrightarrow H_*(THH(A))$$

where HH_* stands for Hochschild homology of an associative ring. This is usually called the Bökstedt spectral sequence.

Associative and commutative monoids. The reformulations (2) and (3) give rise to two spectral sequences. For an associative ring spectrum A we have:

$$\mathrm{Tor}_*^{\mathrm{Ord}\Sigma}(H_*(X), H_*(\underline{A})) \Longrightarrow H_*(THH(A))$$

For a commutative ring spectrum:

$$\mathrm{Tor}_*^{\mathrm{FinSet}}(H_*((S^1)^{\times -}), H_*(\underline{A})) \Longrightarrow H_*(THH(A))$$

A simple application. Let us compute $H_*(THH(MU))$. The Bökstedt spectral sequence gives us:

$$HH_*(H\mathbb{Z}_*(MU)) \Longrightarrow H\mathbb{Z}_*(THH(MU))$$

Observing that $H\mathbb{Z}_*(MU) = \mathbb{Z}[(x_i)_{i>0}]$ (with $|x_i| = 2i$) we get:

$$HH_*(H\mathbb{Z}_*(MU)) = H\mathbb{Z}_*(MU) \otimes \Lambda_{\mathbb{Z}}((x_i)_{i>0})$$

The spectral sequence for FinSet over the rationals collapses since the E_2-term is concentrated on the 0^{th}-column. The E_2-term is $H\mathbb{Q}_*(MU) \otimes \Lambda_{\mathbb{Z}}((x_i)_{i>0})$ and therefore $H\mathbb{Q}_*(THH(MU)) = H\mathbb{Q}_*(MU) \otimes \Lambda_{\mathbb{Z}}((x_i)_{i>0})$. Consequently the Bökstedt spectral sequence must collapse and

$$H\mathbb{Z}_*(THH(MU)) = HH_*(H\mathbb{Z}_*(MU))$$

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